Strict closure of rings

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based on the recent works jointly with

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1. Introduction

Let

- S/R an extension of commutative rings
- $\overline{R}$ the integral closure of R in $\mathrm{Q}(R)$.

We define

$$R \subseteq R^* = \{x \in S \mid x \otimes 1 = 1 \otimes x \text{ in } S \otimes_R S\} \subseteq S$$

and we say that

- R is *strictly closed in S*, if $R = R^*$ holds in S.
- R is *strictly closed*, if $R = R^*$ holds in $\overline{R}$.

Notice that

- $(R^*)^* = R^*$ in S
- $R^* \subseteq T^*$ in S for all $R \subseteq T \subseteq S$.

Example 1.1

Let $S = k[X, Y]$ be the polynomial ring over a field k and consider

$$R = k[X^4, XY^3, Y^4].$$

Then

$$R \subseteq R^* = k[X^4, XY^3, X^7Y^5, Y^4] \subseteq \overline{R} = k[X^4, X^3Y, X^2Y^2, XY^3, Y^4]$$

where R^* denotes the strict closure of R in $\overline{R}$.

(Proof) We set $T = k[X^4, XY^3, X^7Y^5, Y^4]$. Consider $a = b = X^4Y^4$ and $c = XY^3$. Then $a, b, c \in R$, $\frac{a}{c} = \frac{b}{c} = X^3Y \in \overline{R}$ and

$$\frac{ab}{c} \otimes 1 = \frac{a}{c} \otimes \left(c \cdot \frac{b}{c}\right) = \left(\frac{a}{c} \cdot c\right) \otimes \frac{b}{c} = 1 \otimes \frac{ab}{c} \quad \text{in} \quad \overline{R} \otimes_R \overline{R}.$$

This shows $X^7Y^5 = \frac{ab}{c} \in R^*$, so that $T \subseteq R^*$.

Note that $\overline{R} = R + R{\cdot}X^3Y + R{\cdot}X^2Y^2$. We consider a presentation of $\overline{R}$

$$R^{\oplus 5} \xrightarrow{\mathbb{M}} R^{\oplus 3} \xrightarrow{\varepsilon} \overline{R} \longrightarrow 0$$

as an R-module, where $\varepsilon = \begin{bmatrix} 1 & X^3Y & X^2Y^2 \end{bmatrix}$ and

$$\mathbb{M} = \begin{pmatrix} X^2Y^6 & X^4Y^4 & X^3Y^9 & 0 & 0 \\ 0 & -XY^3 & -Y^8 & Y^4 & X^3Y^9 \\ -Y^4 & 0 & 0 & -XY^3 & -X^4Y^8 \end{pmatrix}.$$

By applying $\overline{R} \otimes_R (-)$, we have

$$\begin{array}{ccccccc} \overline{R} \otimes_R R^{\oplus 5} & \xrightarrow{\overline{R}\otimes\mathbb{M}} & \overline{R} \otimes_R R^{\oplus 3} & \xrightarrow{\overline{R}\otimes\varepsilon} & \overline{R} \otimes_R \overline{R} & \longrightarrow & 0. \\ \uparrow \cong & & \uparrow \cong & & & & \\ \overline{R}^{\oplus 5} & \xrightarrow{\mathbb{M}} & \overline{R}^{\oplus 3} & & & & \end{array}$$

Let $\alpha \in R^*$ and write $\alpha = \alpha_0 + \alpha_1 X^3Y + \alpha_2 X^2Y^2$ for $\exists\ \alpha_i \in R$. Set $\beta = \alpha_1 X^3Y + \alpha_2 X^2Y^2$. Then, since $R \subseteq R^*$, we have $\beta \in R^*$.

Therefore

$$\beta \otimes 1 - 1 \otimes \beta = \beta \otimes 1 - \left[\alpha_1(1 \otimes X^3 Y) + \alpha_2(1 \otimes X^2 Y^2)\right] = 0 \text{ in } \overline{R} \otimes_R \overline{R}.$$

By setting $\{e_i\}_{0 \le i \le 2}$ the standard basis of $R^{\oplus 3}$, we obtain

$$\beta \otimes e_0 - [\alpha_1(1 \otimes e_1) + \alpha_2(1 \otimes e_2)] \in \mathrm{Ker}(\overline{R} \otimes \varepsilon)$$

which yields

$$\begin{pmatrix} \beta \\ -\alpha_1 \\ -\alpha_2 \end{pmatrix} \in \mathrm{Im}(\overline{R}^{\oplus 5} \xrightarrow{\mathbb{M}} \overline{R}^{\oplus 3}).$$

This implies $\beta \in J$, the ideal of $\overline{R}$ generated by all the entries of the first row of $\mathbb{M}$. Hence

$$\alpha = \alpha_0 + \beta \in R + (X^2Y^6, X^4Y^4, X^3Y^9)\overline{R} = R + R{\cdot}X^7Y^5 \subseteq T.$$

This shows $T = R^*$. □

Theorem 1.2

Suppose $S = R + Rf_1 + \cdots + Rf_n$, where $n > 0$ and $f_i \in S$ for $1 \le i \le n$. Moreover, we assume that S has a presentation

$$R^{\oplus q} \xrightarrow{\mathbb{M}} R^{\oplus(n+1)} \xrightarrow{\varepsilon} S \longrightarrow 0$$

of R-modules, where $q > 0$ and $\varepsilon = [1 \ f_1 \ \cdots \ f_n]$. Then

$$R^* \subseteq R + J \quad \text{in} \quad S$$

where J denotes an ideal of S generated by all the entries of the first row of $\mathbb{M}$.

Example 1.3

Let $R = k[X^5, XY^4, Y^5]$. Then

$$R^* = k[X^5, X^9Y^6, X^8Y^7, X^4Y^{11}, XY^4, Y^5].$$

Corollary 1.4

Suppose $f_i f_j \in R$ for $1 \le \forall i, j \le n$. Then R is strictly closed in S.

(Proof) Since $S = R + \sum_{i=1}^n Rf_i$, we have $R : S = \bigcap_{i=1}^n [R : f_i]$. Let $\alpha \in \operatorname{Ker} \varepsilon$ and write $\alpha = \sum_{i=0}^n \alpha_i e_i$, where $\alpha_i \in R$. Then

$$\alpha_0 = -\sum_{i=1}^n \alpha_i f_i \quad \text{in} \quad S$$

whence

$$\alpha_0 f_j = -\sum_{i=1}^n (\alpha_i f_i) f_j = -\sum_{i=1}^n \alpha_i (f_i f_j) \in R \quad \text{for} \quad 1 \le \forall j \le n.$$

Thus $\alpha_0 \in R : S$. Hence $J \subseteq R : S \subseteq R$, so that $R^* \subseteq R + J \subseteq R$. □

Example 1.5

Let $S = k[X, Y]$ be the polynomial ring over a field k. Let $n \geq 6$ be an integer and set

$$R = k[X^{n-i}Y^i \mid 0 \leq i \leq n, \ i \neq 1, 3].$$

Then R is a strictly closed Cohen-Macaulay ring with $\dim R = 2$.

(Proof) This follows from $\overline{R} = R + RX^{n-1}Y + RX^{n-3}Y^3$ and $(X^{n-1}Y)^2, (X^{n-1}Y)(X^{n-3}Y^3), (X^{n-3}Y^3)^2 \in R$. □

Example 1.6

Let $(R, \mathfrak{m})$ be a RLR with $\dim R = 2$. Let $\mathfrak{m} = (x, y)$, $I = (x^3, xy^4, y^5)$. Then the Rees algebra

$$\mathcal{R}(I) = R[It]$$

is strictly closed, where t is an indeterminate.

(Proof) We set $J = (x^3, x^2y^2, xy^4, y^5)$. Then $\overline{\mathcal{R}} = R[Jt]$, $\overline{\mathcal{R}} = \mathcal{R}[x^2y^2t]$, and $(x^2y^2t)^2 \in \mathcal{R}$ □

Let $R \subseteq T \subseteq S$. Then $R \subseteq R_T^* \subseteq R_S^* \subseteq S$. Hence

if R is strictly closed in S, then it is strictly closed in T.

Example 1.7

Let $S = k[[t]]$ be the formal power series ring over a field k. Consider

$$R = k[[t^3, t^8, t^{13}]] \subseteq T = k[[t^3, t^5]] \subseteq S.$$

Then R is NOT strictly closed in $S = \overline{R}$, but it is strictly closed in T.

- In 1949, Cahit Arf explored the multiplicity sequences of curve singularities.
- In 1971, J. Lipman defined "Arf rings" for one-dimensional CM semi-local rings.

Definition 1.8 (Lipman, 1971)

Let R be a CM semi-local ring with $\dim R = 1$. Then R is called *an Arf ring*, if the following hold:

(1) Every integrally closed *open* ideal I has a principal reduction.

(2) If $x, y, z \in R$ s.t.

$$x \text{ is a NZD on } R \text{ and } \frac{y}{x}, \ \frac{z}{x} \in \overline{R},$$

then $yz/x \in R$.

Notice that

(1) $I^{n+1} = aI^n$ for $\exists\ n \geq 0$ and $\exists\ a \in I$.

(2) Stability of I (if reduction exists).

Hence

Theorem 1.9 (Lipman, 1971)

Let R be a CM semi-local ring with $\dim R = 1$. Then

$$R \text{ is Arf} \iff \text{Every integrally closed open ideal is stable.}$$

When R is a CM local ring with $\dim R = 1$,

if R is an Arf ring, then R has minimal multiplicity.

We assume

- $(R, \mathfrak{m})$ is a Noetherian complete local domain with $\dim R = 1$
- $R/\mathfrak{m}$ is an algebraically closed field of characteristic 0

Lipman proved:

R is *saturated* $\Longrightarrow$ R has minimal multiplicity.

Moreover, among all Arf rings between R and $\overline{R}$,

$\exists$ the smallest one $\mathrm{Arf}(R)$, called Arf closure.

Lipman extends the results of C. Arf about multiplicity sequences.

This fact leads to obtain a characterization of Arf rings by means of the semigroup of values, which gives rise to the notion of Arf semigroups.

Theorem 1.10

Let $R = k[[H]]$ be the numerical semigroup ring over a field k. Then TFAE.

(1) *R is an Arf ring.*

(2) *R is a weakly Arf ring, i.e., if $x, y, z \in R$ s.t.*

$$x \text{ is a NZD on } R \text{ and } \frac{y}{x}, \ \frac{z}{x} \in \overline{R},$$

then $yz/x \in R$.

(3) *If $x, y, z \in H$ such that $x \leq y$ and $x \leq z$, then $y + z - x \in H$.*

Proposition–Definition 1.11

Let R be a CM semi-local ring with $\dim R = 1$. Suppose $\overline{R}$ is a finitely generated R-module. Then, among all Arf rings between R and $\overline{R}$, there is the smallest Arf ring $\mathrm{Arf}(R)$, called the Arf closure of R.

Conjecture 1.12 (Zariski, 1971)

Let R be a CM semi-local ring with $\dim R = 1$. Suppose $\overline{R}$ is a finitely generated R-module. Then the equality

$$\mathrm{Arf}(R) = R^*$$

holds in $\overline{R}$.

- Zariski's conjecture holds if R contains a field (Lipman).

Theorem 1.13 (Main result)

Zariski's conjecture holds.

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Theorem 1.14

Let $R = k[[H]]$ be the numerical semigroup ring over a field k. Then TFAE.

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$$x \text{ is a NZD on } R \text{ and } \frac{y}{x}, \ \frac{z}{x} \in \overline{R},$$

then $yz/x \in R$.

(3) *H is an Arf semigroup, i.e., if $x, y, z \in H$ such that $x \leq y$ and $x \leq z$, then $y + z - x \in H$.*

(4) *R is strictly closed in $\overline{R}$.*

Example 1.15

$k[[t^n, t^{n+1}, \dots, t^{2n-1}]]$ $(n \geq 2)$ is strictly closed, but $k[[t^3, t^8, t^{13}]]$ is NOT.

H is Arf, if $x, y, z \in H$ such that $x \leq y$ and $x \leq z$, then $y + z - x \in H$.

Let $H = \langle 3, 8, 13 \rangle$. Then

0	1	2
3	4	5
6	7	8
9	10	11
12	13	14
15	16	17

Take $x = 6$, $y = 8$, and $z = 8$. Then $y + z - x = 10 \notin H$, so that H is not Arf. Hence $k[[H]]$ is not strictly closed. $\square$

2. Proof of Zariski's conjecture

Theorem 2.1

Let R be a CM semi-local ring with $\dim R = 1$. Then TFAE.

(1) *R is a strictly closed ring.*

(2) *R is an Arf ring.*

known results

Let R be a CM semi-local ring with $\dim R = 1$. Then

- R is strictly closed $\Longrightarrow$ R is Arf. (Zariski)
- The converse holds if R contains a field. (Lipman)

Proof of $(2) \Rightarrow (1)$

There is a filtration:

$$R \subseteq J : J \subseteq J^2 : J^2 \subseteq \cdots \subseteq J^m : J^m \subseteq \cdots \subseteq \overline{R}$$

where J denotes the Jacobson radical of R. Define

$$R \ \subseteq \ R^J = \bigcup_{m \geq 0} [J^m : J^m] \ \subseteq \ \overline{R}.$$

For $n \geq 0$, we set

$$R_n = \begin{cases} R & \text{if } n = 0 \\ R_{n-1}^{J(R_{n-1})} & \text{if } n \geq 1 \end{cases}$$

where $J(R_{n-1})$ stands for the Jacobson radical of R_{n-1}.

Hence

$$R \subseteq R_1 \subseteq \cdots \subseteq R_n \subseteq \cdots \subseteq \overline{R}.$$

Step 1

The equality $\overline{R} = \bigcup_{n \geq 0} R_n \ (= \varinjlim R_n)$ holds.

Step 2

The equality $R = R^*$ holds in R_n for $\forall n \geq 0$.

Lemma 2.2 (Key lemma)

Let $(R, \mathfrak{m})$ be a CM local ring with $\dim R = 1$. Suppose that $\mathfrak{m}^2 = z\mathfrak{m}$ for some $z \in \mathfrak{m}$. Let $R_1 \subseteq C \subseteq \overline{R}$ be an intermediate ring s.t. C is a finitely generated R-module and let

$$\alpha : C \otimes_R C \to C \otimes_{R_1} C$$

be an R-algebra map s.t. $\alpha(x \otimes y) = x \otimes y$ for $\forall x, y \in C$. Then

$$\operatorname{Ker} \alpha = (0) :_{C \otimes_R C} z$$

holds.

Let $x \in R^*$ in $\overline{R}$ and choose $n \geq 0$ such that $x \in R_n$. Since $\overline{R} = \varinjlim R_m$, we get

$$\begin{array}{rcl} \overline{R} \otimes_R R_n & \to & \overline{R} \otimes_R \overline{R} = \varinjlim (\overline{R} \otimes_R R_m) \\ x \otimes 1 - 1 \otimes x & \mapsto & 0. \end{array}$$

There exists $\ell \geq n$ such that

$$\overline{R} \otimes_R R_n \to \overline{R} \otimes_R R_\ell, \quad x \otimes 1 - 1 \otimes x \mapsto 0.$$

Since

$$\begin{array}{rcl} R_n \otimes_R R_\ell & \to & \overline{R} \otimes_R R_\ell = \varinjlim (R_m \otimes_R R_\ell) \\ x \otimes 1 - 1 \otimes x & \mapsto & 0, \end{array}$$

there exists $p \geq n$ such that

$$R_n \otimes_R R_\ell \to R_p \otimes_R R_\ell, \quad x \otimes 1 - 1 \otimes x \mapsto 0.$$

For $q \in \mathbb{Z}$ such that $q \geq p$ and $q \geq \ell$, we obtain

$$\begin{array}{ccccc} R_n \otimes_R R_\ell & \to & R_p \otimes_R R_\ell & \to & R_q \otimes_R R_q \\ x \otimes 1 - 1 \otimes x & \mapsto & 0 & \mapsto & 0 \end{array}$$

Therefore

$$x \in R_n \subseteq R_q \quad \text{and} \quad x \otimes 1 = 1 \otimes x \quad \text{in } R_q \otimes_R R_q$$

so that $x \in R^*$ in R_q. Thus $x \in R$. Hence $R = R^*$ in $\overline{R}$. □

Theorem 2.3

Let R be a CM semi-local ring with $\dim R = 1$. Then

$$R \text{ is strictly closed} \iff R \text{ is Arf.}$$

Hence, $\mathrm{Arf}(R) = R^*$ holds, provided $\overline{R}$ is a finitely generated R-module.

A commutative ring R is said to be *weakly Arf*, provided

$$yz/x \in R, \text{ whenever } x, y, z \in R \text{ s.t. } x \in R \text{ is a NZD, } y/x, z/x \in \overline{R}.$$

Theorem 2.4

Let R be a Noetherian ring with (S_2). Then TFAE.

(1) *R is strictly closed.*

(2) *R is weakly Arf, and R_P is Arf for $\forall P \in \operatorname{Spec} R$ with $\operatorname{ht}_R P = 1$.*

Corollary 2.5

Let $(R, \mathfrak{m})$ be a Noetherian local ring with $\dim R \geq 2$ and (S_2). Then

$$R \text{ is strictly closed} \iff R \text{ is weakly Arf.}$$

Theorem 2.6

Let B be a CM semi-local ring with $\dim B = 1$. Let A be a subring of B. We assume that B is integral over A and A is a direct summand of B as an A-module. If B is an Arf ring, then so is A.

Corollary 2.7

Let R be a CM semi-local ring with $\dim R = 1$. Then

$$R \text{ is Arf} \implies R^G \text{ is Arf}$$

for every finite subgroup G of $\operatorname{Aut} R$ s.t. the order of G is invertible.

3. Strictly closed rings

Question 3.1

What kind of rings are strictly closed?

Theorem 3.2

Let R be a commutative ring and T an R-subalgebra of $\mathrm{Q}(R)$. Let V be a non-empty subset of T s.t. $T = R[V]$. If $fg \in R$ for all $f, g \in V$, then R is strictly closed in T.

Corollary 3.3

Let R be a commutative ring and $J = (a_1, a_2, \ldots, a_n)$ $(n \geq 3)$ an ideal of R s.t. $a_1^2 = a_2 a_3$. Set $I = (a_2, a_3, \ldots, a_n)$ and consider

$$\mathcal{R} = \mathcal{R}(I) \subseteq \mathcal{T} = \mathcal{R}(J)$$

Then $\mathcal{R}$ is strictly closed in $\mathcal{T}$, provided I contains a NZD on R.

Theorem 3.4

The Stanley-Reisner ring $R = k[\Delta]$ of Δ is strictly closed.

Let

- R a Noetherian reduced ring.
- $\operatorname{Min} R = \{\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_\ell\}$, where $\ell = \sharp \operatorname{Min} R \geq 2$.

We assume

$$(*) \qquad R/\mathfrak{p}_i \text{ is integrally closed for } 1 \leq \forall i \leq \ell.$$

Then

$$0 \to R \xrightarrow{\varphi} R/\mathfrak{p}_1 \oplus R/\mathfrak{p}_2 \oplus \cdots \oplus R/\mathfrak{p}_\ell = \overline{R}$$

where $\varphi(x) = (\overline{x}, \overline{x}, \ldots, \overline{x})$ for $x \in R$. Hence

$$\overline{R} = \sum_{i=1}^{\ell} Re_i = \bigoplus_{i=1}^{\ell} Re_i$$

and

$$\overline{R} \otimes_R \overline{R} = \sum_{1 \leq i,j \leq \ell} R(e_i \otimes e_j) = \bigoplus_{1 \leq i,j \leq \ell} Re_i \otimes_R Re_j.$$

where $e_i = (0, \ldots, 0, \overset{\overset{i}{\vee}}{1}, 0, \ldots, 0) \in \overline{R}$ for $1 \leq i \leq \ell$.

Let $x \in \overline{R}$ and write $x = (\overline{x_1}, \overline{x_2}, \ldots, \overline{x_\ell})$ with $x_i \in R$. Then

$$x \otimes 1 = \sum_{1 \le i,j \le \ell} x_i(e_i \otimes e_j) \quad \text{and} \quad 1 \otimes x = \sum_{1 \le i,j \le \ell} x_j(e_i \otimes e_j).$$

Therefore

$$x \otimes 1 = 1 \otimes x \iff x_i(e_i \otimes e_j) = x_j(e_i \otimes e_j) \quad \text{for} \quad 1 \le \forall i, j \le \ell.$$

Since

$$R(e_i \otimes e_j) = Re_i \otimes_R Re_j \cong R/\mathfrak{p}_i \otimes_R R/\mathfrak{p}_j \cong R/[\mathfrak{p}_i + \mathfrak{p}_j],$$

we have

$$x \otimes 1 = 1 \otimes x \iff x_i - x_j \in \mathfrak{p}_i + \mathfrak{p}_j \quad \text{for} \quad 1 \le \forall i, j \le \ell.$$

Proposition 3.5

Suppose that $\ell = 2$. Then R is strictly closed.

Proof.

Let $x \in \overline{R}$ and assume that $x \otimes 1 = 1 \otimes x$ in $\overline{R} \otimes_R \overline{R}$. Write $x = (\overline{x_1}, \overline{x_2})$ with $x_i \in R$. Since $x_1 - x_2 \in \mathfrak{p}_1 + \mathfrak{p}_2$, we get

$$x_1 - x_2 = y_1 + y_2$$

for some $y_i \in \mathfrak{p}_i$. Because

$$x = (\overline{x_1}, \overline{x_2}) = (\overline{x_2 + y_1 + y_2}, \overline{x_2}) = (\overline{x_2 + y_2}, \overline{x_2}) = (\overline{x_2 + y_2}, \overline{x_2 + y_2}),$$

we have $x \in R$, whence $R = R^*$ in $\overline{R}$. □

Corollary 3.6

Let S be a RLR and let $a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_n$ be a regular system of parameters of S. Then

$$R = S/[(a_1, a_2, \ldots, a_m) \cap (b_1, b_2, \ldots, b_n)]$$

is a strictly closed ring.

Thank you for your attention.